ADVANCES IN GROUP RINGS†

BY

D. S. PASSMAN

CONTENTS

1. Introduction

Let K be a field and let G be a multiplicative group. Then the group ring $K[G]$ of G over K is a K-algebra with the elements of G as a basis. To be more precise, $K[G]$ consists of all formal finite sums $\alpha = \sum a_x x$ with $x \in G$ and $a_x \in K$. Moreover, addition is given componentwise and multiplication is defined distributively using the group multiplication of G. Finally, the identification of $x \in G$ with $1 \cdot x \in K[G]$ yields a natural embedding of G into this ring. It is clear that these easily defined group rings offer rather attractive objects of study. Furthermore, as the name implies, this subject is a meeting place for two essentially different disciplines and indeed the results are frequently a rather nice blending of group theory and ring theory. In this paper we describe the present state of the subject of infinite group rings and, while we make no claim of completeness, we hope at east to touch upon each of the major problems.

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If G is finite then $K[G]$ can be studied using trace functions and the fairly strong structure theorems for finite dimensional algebras. On the other hand, if G is infinite then these methods are no longer available and the problems are therefore correspondingly more difficult. Modulo a few exceptions, the first papers on infinite group rings appeared in the early 1950s. Important impetus was given to this subject by the inclusion of group ring problems in Kaplansky's Ram's Head Inn problems (1957 [51], 1970 [53]) and by the inclusion of group ring material in the books of Lambek (1966 [55]), Ribenboim (1969 [83]) and the monograph of Herstein (1971 [40]). The only books devoted totally to this subject are those of Passman (1971 [69]), Mihalev and Zalesskii (1973 [60]), and Bovdi 1974 [105)]. Since group rings are rings after all, the questions we ask about them must necessarily be ring theoretic in nature. On the other hand, the answers and the techniques involved usually exhibit a strong group theoretic flavor.

There are obviously a number of generalizations and analogs of $K[G]$. For example we may clearly form the group ring $R[G]$ over any coefficient ring R and we can also form certain twisted group rings and crossed products. While the study of crossed products is really another subject entirely, results on the other generalizations frequently follow immediately from $K[G]$ or at least by quite similar techniques. Thus the important case is that of group rings over fields. In addition there are the analytic analogs $L^1(G)$ and $W(G)$ but again these are really different subjects in which different sorts of problems are considered. Nevertheless, by way of the obvious inclusion $C[G] \subseteq L^1(G) \subseteq W(G)$, analytic results have motivated and also proved theorems about the complex group ring $C[G]$. Since most of these theorems can now be obtained algebraically we will not pursue further these interrelations here.

We start now by making a few simple observations. Suppose H is a subgroup of G. Then the inclusion $H \subseteq G$ gives rise to the obvious inclusion $K[H] \subseteq K[G]$. In fact if $\{x_v\}$ is a set of left coset representatives for H in G then $K[G] = \sum x_v K[H]$ exhibits $K[G]$ as a free right $K[H]$-module. Furthermore, there is a natural projection map $\pi_H: K[G] \to K[H]$ given by

$$\pi_H\left(\sum_{x \in G} a_x x\right) = \sum_{x \in H} a_x x.$$

This map is by no means a ring homomorphism but it is a $K[H]$-module homomorphism with the following property.

LEMMA 1.1. *If I is an ideal of $K[G]$ then $\pi_H(I)$ is an ideal of $K[H]$ and $I \subseteq \pi_H(I) \cdot K[G]$.*

Moreover, for H normal in G, we say that H controls the ideal I if $I = \pi_H(I) \cdot K[G]$ or, equivalently, if $I = (I \cap K[H]) \cdot K[G]$.

Suppose again that H is normal in G. Then the quotient map $G \to G/H$ extends naturally to the epimorphism $\rho_H \colon K[G] \to K[G/H]$. In the special case when $H = G$ the kernel of this map is $\omega(K[G])$, the augmentation ideal of $K[G]$ or, in other words, the set of all elements of $K[G]$ with coefficient sum zero. In general the kernel of ρ_H is $\omega(K[H]) \cdot K[G]$, the augmentation ideal of $K[H]$ in $K[G]$.

Now let $\alpha = \Sigma a_x x \in K[G]$. Then we define the support of α to be

$$\operatorname{supp} \alpha = \{x \in G \mid a_x \neq 0\}.$$

Thus supp α is a finite subset of G which is nonempty for $\alpha \neq 0$. In discussing elements of G we let $o(x)$ denote the order of x and we say that x is a p-element if $o(x) = p^n$ for some $n \geqq 1$. Similarly x is a p'-element if $o(x)$ is finite and prime to p and G is said to be a p'-group if it contains no p-elements. Finally we remark that the map $\Sigma a_x x \to \Sigma a_x x^{-1}$ defines an anti-automorphism of $K[G]$ and thus the group ring enjoys similar right and left properties.

2. Algebraic elements

In the study of algebraic elements in $K[G]$ we are mainly concerned with properties of nilpotent and idempotent elements. Suppose that G is a finite group and K is a field of characteristic 0. If $s \colon K[G] \to K$ denotes the trace of the regular representation of $K[G]$ then it is easy to see that $s(\Sigma a_x x) = |G| a_1$. Thus if $\alpha = \Sigma a_x x$ is nilpotent then certainly $s(\alpha) = 0$ and hence $a_1 = 0$. On the other hand, if α is an idempotent with $\dim_K K[G]\alpha = r$ then $s(\alpha) = r$ so $a_1 = r/|G|$ is a rational number between 0 and 1. For arbitrary G and K we define $\operatorname{tr} \colon K[G] \to K$ by tr $(\Sigma a_x x) = a_1$. Then, as expected, tr behaves like a trace function and we consider $\operatorname{tr}(\alpha)$ for α nilpotent and idempotent. The starting point is the following elementary

LEMMA 2.1. *Let E be an algebra over a field of characteristic $p > 0$. If $\alpha_1, \alpha_2, \cdots, \alpha_m \in E$ then*

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_m)^{p^n} = \alpha_1^{p^n} + \alpha_2^{p^n} + \cdots + \alpha_m^{p^n} + \beta$$

with $\beta \in [E, E]$.

Now if $E = K[G]$, then it is easy to see that the Lie ideal $[E, E]$ consists of all

$\beta = \Sigma b_x x$ such that the sum of the coefficients over each conjugacy class of G is zero. Hence $\operatorname{tr} \beta = 0$ so $\operatorname{tr}(\alpha_1 + \alpha_2)^p = (\operatorname{tr} \alpha_1)^p + (\operatorname{tr} \alpha_2)^p$ and this yields easily

THEOREM 2.2 (Passman 1962 [64], Connell 1963 [15]). *Let K be a field of characteristic $p > 0$. If $\alpha \in K[G]$ is nilpotent then either* $\operatorname{tr} \alpha = 0$ *or* $\operatorname{supp} \alpha$ *contains a p-element. In particular if G is a p'-group then $K[G]$ has no nonzero nil ideals.*

Now suppose that char $K = 0$ and that $\alpha = \Sigma a_x x$ is nilpotent. Then $\alpha \in R[G]$ where R is the finitely generated integral domain $R = Z[a_x \mid x \in \operatorname{supp} \alpha]$. Since there are only finitely many primes q with $\operatorname{supp} \alpha$ containing a q-element, it follows from the Extension Theorem for Places that there exists a homomorphism $\phi: R \to F$ such that F is a field of characteristic $p > 0$, $\phi(a_1) \neq 0$ if $a_1 \neq 0$ and $\operatorname{supp} \alpha$ contains no p-elements. Since ϕ extends naturally to $\phi: R[G] \to F[G]$ we immediately conclude from the above

COROLLARY 2.3. *Let K have characteristic* 0. *If $\alpha \in K[G]$ is nilpotent then* $\operatorname{tr} \alpha = 0$. *Hence $K[G]$ has no nonzero nil ideals.*

The corresponding results for idempotents are more difficult. We start again with char $K = p > 0$ and define the nth trace tr_n by

$$\operatorname{tr}_n(\Sigma a_x x) = \Sigma_{o(x) = p^n} a_x.$$

Thus $\operatorname{tr}_0 = \operatorname{tr}$ and it follows easily from Lemma 2.1 that

$$\operatorname{tr}_n(\alpha^p) = (\operatorname{tr}_{n+1}\alpha)^p \text{ for } n > 0$$

$$\operatorname{tr}_0(\alpha^p) = (\operatorname{tr}_1\alpha)^p + (\operatorname{tr}_0\alpha)^p.$$

Now if α is an idempotent then $\alpha^p = \alpha$ and also certainly $\operatorname{tr}_n\alpha = 0$ for all sufficiently large n. Thus by above $\operatorname{tr}_n\alpha = 0$ for all $n \geqq 1$ and $\operatorname{tr}_0 \alpha = (\operatorname{tr}_0\alpha)^p$. This yields the following beautiful

THEOREM 2.4 (Zalesskii 1972 [102]). *Let* char $K = p > 0$ *and let $e \in K[G]$ be an idempotent. Then* $\operatorname{tr} e \in GF(p)$.

The analogous result in characteristic 0 follows as before by a place argument. Namely we have $e \in R[G]$ and we show that if $\operatorname{tr} e \notin Q$, the rationals, then there exists a homomorphism $\phi: R \to F$ where F is a field of chacateristic $p > 0$ and $\phi(\operatorname{tr} e) \notin GF(p)$. This is fairly routine if $\operatorname{tr} e$ is assumed transcendental over Q, but if $\operatorname{tr} e$ is algebraic then the possible images ϕ ($\operatorname{tr} e$) are quite restricted. Fortunately in this case we can apply the Frobenius Density Theorem (1896 [32]), a result on algebraic number fields, to deduce the following conjecture of Kaplansky.

COROLLARY 2.5 (Zalesskii 1972 [102]). *Let* char $K = 0$ *and let* $e \in K[G]$ *be an idempotent. Then* tr e *is rational.*

Furthermore we have

THEOREM 2.6 (Kaplansky 1969 [52]). *Let* char $K = 0$ *and let* $e \in K[G]$ *be an idempotent. Then* $0 < \operatorname{tr} e \leqq 1$ *with* $\operatorname{tr} e = 1$ *if and only if* $e = 1$.

The proof of this result is analytic in nature. Kaplansky works in $W(G)$ and Montgomery (1969 [61]) in the uniform closure of $C[G]$. On the other hand, a later proof of Passman (1971 [70]) uses limits but stays within $C[G]$. Other properties of idempotents can be found in Bovdi-Mihovski (1973 [11]). Now a ring is said to be von Neumann finite if $\alpha\beta = 1$ implies $\beta\alpha = 1$. Suppose $\alpha, \beta \in K[G]$ with $\alpha\beta = 1$. Then $e = \beta\alpha$ is an idempotent and $\operatorname{tr} e = \operatorname{tr} \beta\alpha = \operatorname{tr} \alpha\beta = 1$. Thus $e = 1$ by the above and we have

COROLLARY 2.7 (Kaplansky 1969 [52]). *If* char $K = 0$ *then* $K[G]$ *is von Neumann finite.*

The corresponding question in characteristic $p > 0$ is open.

Much more can, of course, be said if e is central. Indeed, by using some later techniques one can obtain the following theorem whose second part is a result of Osima (1955 [63]).

THEOREM 2.8. *Let* e *be a central idempotent in* $K[G]$. *Then* $\langle \operatorname{supp} e \rangle$ *is a finite normal subgroup of* G. *Moreover if* char $K = p > 0$ *then* supp e *consists of* p'*-elements.*

Finally we mention two results of Formanek. The first uses a modification of the Zalesskii trace argument and then applies Theorem 2.6 to conclude

THEOREM 2.9 (Formanek 1973 [26]). *Let* G *be a group with the property that* $x \in G$ *can be conjugate to* x^n *only when* $n = \pm 1$. *If* K *is a field of characteristic* 0 *then* $K[G]$ *has no nonidentity idempotents.*

Observe that the above assumption on G implies that G is torsion free. Conversely if G is torsion free and satisfies the ascending chain condition on cyclic subgroups then G is easily seen to satisfy the assumption. Thus the above theorem applies to a large class of torsion-free groups.

The second result concerns group rings in which all elements are algebraic. It was shown by Herstein (1968 [39]) that if char $K = 0$ then $K[G]$ is algebraic if and only if G is locally finite. This has been extended to

THEOREM 2.10 (Formanek [27]). *Let* char $K = 0$. *Then* G *is locally finite if* *and only if every element of $K[G]$ is either right invertible or a left zero divisor.*

Again the corresponding question in characteristic $p > 0$ is open. Finally some related results of interest appear in (Farkas 1973 [21]).

3. Linear identities

A linear identity is an equation in $K[G]$ of the form $\alpha_1 x\beta_1 + \alpha_2 x\beta_2 + \cdots + \alpha_n x\beta_n = 0$ which holds for all $x \in G$ or perhaps for just a large subset of the group. Such identities arise in a number of different contexts and are therefore worthy of study. Perhaps the simplest example is the equation $x\alpha - \alpha x = 0$ for all $x \in G$ which, of course, merely says that α is central in $K[G]$. If $y \in \operatorname{supp} \alpha$ here then clearly

$$y^x = x^{-1}yx \in \operatorname{supp}\, x^{-1}\alpha x = \operatorname{supp} \alpha.$$

Thus since supp α is a finite set, it follows that there are only a finite number of distinct y^x with $x \in G$. The set of all such elements $y \in G$ with this property is therefore of interest. We define

$$\Delta = \Delta(G) = \{x \in G \mid [G : C_G(x)] < \infty\}.$$

It is quite easy to see that Δ is a characteristic subgroup of G, the so-called fc subgroup of G, and it has the following nice properties (Neumann 1951 [62]).

LEMMA 3.1. *The set of torsion elements of $\Delta(G)$ forms a characteristic subgroup $\Delta^+(G)$ and Δ/Δ^+ is torsion free abelian. Furthermore, any finite subset of Δ^+ is contained in a finite normal subgroup of G.*

Let $\theta = \pi_\Delta$ denote the projection map $\theta\colon K[G] \to K[\Delta(G)]$ and let $\theta^+ = \pi_{\Delta^+}$. We then have the following important

LEMMA 3.2. *Suppose that*

$$\alpha_1 x\beta_1 + \alpha_2 x\beta_2 + \cdots + \alpha_n x\beta_n = 0$$

is a linear identity in $K[G]$ which holds for all $x \in G$. Then

$$\theta(\alpha_1)\beta_1 + \theta(\alpha_2)\beta_2 + \cdots + \theta(\alpha_n)\beta_n = 0.$$

We remark that this is merely one of a large number of variants which have proved useful in the study of group rings. In particular this yields the following

THEOREM 3.3 (Passman 1962 [64]). *Let A and B be ideals in $K[G]$ with $AB = 0$. Then $\theta(A)\theta(B) = 0$.*

Thus the question of when $K[G]$ is prime or semiprime is immediately reduced to the corresponding question in $K[\Delta(G)]$ which is then easily solved. Indeed we have

THEOREM 3.4 (Connell 1963 [15]). *$K[G]$ is prime if and only if $\Delta(G)$ is torsion-free abelian.*

THEOREM 3.5 (Passman 1962 [64]). *$K[G]$ is semiprime if and only if either* char $K = 0$ *or* char $K = p > 0$ *and $\Delta(G)$ is a p'-group.*

Moreover these easily yield relationships between theprimeness and semi-prime ness of $K[G]$ and of its center $Z(K[G])$. In this light it is interesting to observe that the Jacobson radical of $Z(K[G])$ is always a nil ideal (Passman 1974 [117]).

For additional results it is necessary to take a closer look at finitely generated subgroups of $\Delta(G)$. If H is such a subgroup then it is easy to see that H has a torsion-free central subgroup Z of finite index n. Moreover, since Z is torsion-free abelian it then follows that all nonzero elements of $K[Z]$ are regular in $K[H]$ and thus it is trivial to form the ring of quotients $E = K[Z]^{-1}K[H]$. This is, of course, the set of all formal fractions $\eta^{-1}\alpha$ with $\eta \in K[Z]$, $\eta \neq 0$, $\alpha \in K[H]$ and with the usual identifications made. Thus if $F = K[Z]^{-1}K[Z]$, then F is a central subfield of E and E is an F-algebra of finite dimension n. In fact E is isomorphic to a twisted group ring of H/Z over F. Let us suppose further that K is algebraically closed. Since $H/\Delta^+(H)$ is a finitely generated, torsion-free abelian group it then follows that $\mathscr{H} = \mathrm{Hom}(H/\Delta^+(H), K^\bullet)$ is isomorphic to a finite direct product of copies of the multiplicative group $K^\bullet = K - \{0\}$. Hence $\mathscr{H}$ is a divisible group and has no proper subgroups of finite index. Furthermore, if $\lambda \in \mathscr{H}$, then viewing λ as a homomorphism from H to $K^\bullet$ we can define the automorphism $\tilde{\lambda}$ of $K[H]$ by

$$\tilde{\lambda}(\Sigma a_x x) = \Sigma a_x \lambda(x) x.$$

In this way $\mathscr{H}$ acts on $K[H]$ and then also on E.

As an application let us consider a central idempotent $e \in K[H] \subseteq E$. Now $\mathscr{H}$ permutes the finitely many central idempotents of the finite dimensional algebra E and, since $\mathscr{H}$ has no proper subgroups of finite index, it follows that $\mathscr{H}$ must fix them all. Thus $\tilde{\lambda}(e) = e$ for all $\lambda \in \mathscr{H}$ and this implies immediately that supp $e \subseteq \Delta^+(H)$. In the same way, if E is known to be semisimple, then $\mathscr{H}$ fixes the finitely many ideals of E and from this and Theorem 3.3 we can conclude

THEOREM 3.6 (Passman 1970 [67], M. Smith 1971 [87]). *Let $K[G]$ be semiprime and suppose that A and B are ideals in $K[G]$ with $AB = 0$. Then $\theta^+(A)\theta^+(B) = 0$.*

COROLLARY 3.7 (M. Smith 1971 [87]). *Let $K[G]$ be semiprime and suppose that A is an annihilator ideal in $K[G]$. Then $A = \theta^+(A) \cdot K[G]$.*

Thus $\Delta^+(G)$ controls annihilator ideals. We remark that the above two results are false without the semiprime assumption.

As another application, suppose that $K[G]$ has a classical ring of quotients $Q(K[G])$. If $\alpha^{-1}\beta$ is central in this ring then $\alpha^{-1}\beta$ commutes with $x\alpha$ for all $x \in G$ and we easily obtain the linear identity $\alpha x \beta = \beta x \alpha$. Furthermore, since α is regular it can be shown by the above techniques that there exists $\gamma \in K[G]$ such that $\theta(\gamma\alpha)$ is central and regular in $K[G]$. This then yields.

THEOREM 3.8 (M. Smith 1971 [87], Passman 1972 [72]). *Let $K[G]$ be an order in $Q(K[G])$. Then the center of $K[G]$ is an order in the center of $Q(K[G])$.*

Additional results of this nature can be found in (M. Smith 1973 [88]). We remark that M. Smith's proof of the above in characteristic 0 uses instead of θ the function $\natural$ where $\alpha^\natural$ is the average of the finitely many G-conjugates of $\theta(\alpha)$. This function maps $K[G]$ into its center and agrees with a certain trace function defined on $W(G)$.

There is also the question of the existence of $Q(K[G])$. It follows easily from the Ore conditions and the freeness of $K[G]$ over $K[H]$ that

LEMMA 3.9. *If $K[H]$ has a classical ring of quotients for all finitely generated subgroups H of G, then $K[G]$ has a classical ring of quotients.*

Using this, P. Smith (1971 [90]), Herstein and Small (1971 [41]), Lewin (1972 [57]) and Hughes 1973 [45]) have shown the existence of $Q(K[G])$ for some rather special classes of groups. On the other hand, the ring of quotients does not exist for $G = \langle x, y \rangle$, the free group on two generators, since the Ore condition $\alpha(x + x^{-1}) = \beta(y + y^{-1})$ is easily seen to have no nonzero solution in the group ring.

Finally we remark that the Utumi or maximal quotient ring Q_m of a group ring has also been studied. Of course here there is no problem of the existence of $Q_m(K[G])$ since maximal quotients always exist. Rather, the concern is with the nature of the center of this ring and the following analog of Theorem 3.8 has been obtained.

THEOREM 3.10. (Formanek [30]). *Let $K[G]$ be a group ring with center C. Then, with respect to a certain natural embedding, $Q_m(K[\Delta(G)])$ contains the center Z of $Q_m(K[G])$. Furthermore if $K[G]$ is semiprime then $Q_m(C) = Z$.*

4. Polynomial identities

Let H be a subgroup of G of finite index n. If $V = K[G]$ then, as we have seen, V is a free left $K[H]$-module of rank n and also certainly a faithful right $K[G]$-module. Thus, since right and left multiplication commute as operators on V, it follows that $K[G] \subseteq K[H]_n$, the ring of $n \times n$ matrices over $K[H]$. If in addition H is abelian, then the theorem of Amitsur and Levitzki (1950 [3]) implies that $K[H]_n$ satisfies the standard polynomial identity s_{2n} and we have (Kaplansky 1949 [50], Amitsur 1961 [2]).

LEMMA 4.1. *If G has an abelian subgroup of finite index n, then $K[G]$ satisfies the standard identity s_{2n}.*

In other words, we have certain sufficient conditions for $K[G]$ to satisfy a polynomial identity. One then asks whether these conditions are perhaps also necessary.

The first result in this direction is due to Amitsur (1961 [2]) who verified the necessity in case char $K = 0$ and $K[G]$ satisfies a polynomial identity of degree at most 4. The characteristic 0 case was completed by Isaacs and Passman (1964 [46], 1965 [47]). They showed in fact that if $K[G]$ satisfies a polynomial identity of degree n then G has an abelian subgroup A with $[G: A] \leqq J(n)$, the function associated with Jordan's theorem on finite complex linear groups. The proof involved studying finite groups by character theoretic methods and then deducing the general result by using certain group theoretic reductions.

The first results in characteristic $p > 0$ were obtained by M. Smith (1971 [87]). She showed, for example, that if $K[G]$ is a prime ring which satisfies a polynomial identity of degree n then G has an abelian subgroup A with $[G: A] \leqq (n/2)^2$ and she also considered finitely generated groups. The proofs for the most part were ring theoretic in nature. Suppose that R is a prime ring which satisfies a polynomial identity of degree n. Then, by a theorem of Posner, R has a classical ring of quotients $Q(R)$ which has dimension at most $(n/2)^2$ over its central subfield. This result with $R = K[G]$ is then applied in conjunction with Theorem 3.8 which studies the center of $Q(K[G])$, to deduce the above-mentioned theorem on prime group rings. Furthermore, M. Smith made the following crucial observation. If G has an abelian subgroup A of finite index, then clearly $A \subseteq \Delta(G)$ and Δ has finite index. Thus a first step in finding such an abelian subgroup is to show that $[G: \Delta] < \infty$. This was done in (Passman 1971 [71]) and about a year later the problem was completely solved using combinatorial techniques and linear identities.

The answer is somewhat different than was expected. We say that A is a p-abelian group if A' is a finite p-group.

THEOREM 4.2 (Isaacs-Passman 1964 [46], 1965 [47]). *Let K be a field of characteristic 0. If $K[G]$ satisfies a polynomial identity of degree n then G has an abelian subgroup A with $[G\colon A]$ bounded by a fixed function of n.*

THEOREM 4.3. (Passman 1972 [73]). *Let K be a field of characteristic $p > 0$. If G has a p-abelian subgroup A of finite index, then $K[G]$ satisfies the standard identity of degree $2|A'|\cdot[G\colon A]$. Conversely if $K[G]$ satisfies a polynomial identity of degree n then G has a p-abelian subgroup A with $|A'|\cdot[G\colon A]$ bounded by a fixed function of n.*

THEOREM 4.4 (Passman 1972 [74]). *If $K[G]$ satisfies a polynomial identity of degree n then $[G\colon \Delta(G)] \leqq n/2$ and $|\Delta(G)'| < \infty$.*

We remark that the bound in the latter result is best possible but that of Theorem 4.3 is really astronomical. Fortunately, at least in the case when $|\Delta(G)'|$ is prime to the characteristic of K, and in particular when $K[G]$ is semiprime, more realistic bounds can be found. The proof of this appears in (Passman 1971 [69, Chapt. II]) and is just a reformulation in terms of modules of the original Isaacs-Passman argument. We discuss some aspects of this below.

Suppose G is a finite group and K is an algebraically closed field. We say that $K[G]$ has rb$n$ (representation bound n) if all the irreducible $K[G]$-modules have degree at most n. Thus it is trivial to see that if $K[G]$ satisfies a polynomial identity of degree n then $K[G]$ has rb$(n/2)$. The proof of the semiprime result is formulated in terms of representation bounds, and the intrusion of polynomial identities into the arguments of Passman (1971 [69]) is confusing. Fortunately it is also unnecessary in view of the following.

LEMMA 4.5. *Let $K[G]$ have* rb$n$ *with $G \neq \langle 1\rangle$. In addition, if* char $K = p > 0$, *assume that G' is a p'-group. Then G has a non-identity conjugacy class of size $\leqq n^2$.*

PROOF. We consider only the more difficult characteristic $p > 0$ situation. By assumption, $G = HP$ where H is a normal p-complement and P is an abelian Sylow p-subgroup and we may clearly assume that $H \neq \langle 1\rangle$. Let $n_0 = 1, n_1, \cdots, n$.

be the degrees of the irreducible $K[G]$-modules with n_0 corresponding to the principal module. It then follows from the above and from a good deal of work that

$$K[G] \simeq (R_0)_{n_0} + (R_1)_{n_1} + \cdots + (R_t)_{n_t}$$

where $R_i = K[P_i]$ for a suitable subgroup $P_i \subseteq P$ and $R_0 = K[P]$. Therefore each R_i is commutative. If $s = \sum_{i=1}^{t} \dim_K R_i$ then since $K[G]$ has rb$n$ we have

$$\dim_K \text{center } K[G] = \sum_{i=0}^{t} \dim_K R_i = |P| + s$$

and

$$|G| = \dim_K K[G] = \sum_{i=0}^{t} (\dim_K R_i)n_i^2 \leqq |P| + sn^2.$$

Now the center of $K[G]$ is spanned by the conjugacy class sums so G has precisely $|P| + s$ such classes. Furthermore, since $G = HP$ it follows that precisely $|P|$ of these classes intersect P and if we denote these by $D_1, D_2, \cdots, D_{|P|}$ and the remaining ones by $C_1, C_2, \cdots, C_s$ then we have clearly

$$|G| = \sum_{i=1}^{|P|} |D_i| + \sum_{i=1}^{s} |C_i| \geqq |P| + sc$$

where $c = \min |C_i|$. Finally, since $H \neq \langle 1 \rangle$ we have $s > 0$ and the latter two equations combine to yield $n^2 \geqq c$.

The improved bound which can be obtained by means of the above is as follows.

THEOREM 4.6. *Let $K[G]$ be semiprime and satisfy a polynomial identity of degree n. If $m = [n/2]$, then G has an abelian subgroup A with $[G: A] \leqq (m!)^m$.*

Finally, there remains the question of whether a group ring which satisfies a polynomial identity can always be embedded in the full matrix ring over some commutative K-algebra. This is certainly the case if G has an abelian subgroup of finite index but there is no general answer as yet.

Note also that generalized polynomial identities were considered in (Passman 1971 [116]) and that *p.i.* direct summands were studied by Formanek [106] and Passman (1974 [118]).

5. Semisimplicity (general results)

Probably the most difficult and exciting of all the group ring problems is the one on semisimplicity. Namely, we wish to find when $K[G]$ is semisimple and, perhaps more generally, to determine the structure of the Jacobson radical $JK[G]$.

This problem has been studied for over twenty years and we are only now approaching the situation of having a viable conjecture.

Suppose first that char $K = 0$. Then by Corollary 2.3, $K[G]$ has no nonzero nil ideals and presumably it is also semisimple. Observe that if F is a field extension of K then $F \otimes_K K[G] = F[G]$ so results on the behavior of the radical under field extensions apply nicely here to yield

THEOREM 5.1 (Amitsur 1959 [1]). *Let K be a field of characteristic* 0 *which is not an algebraic extension of the rationals. Then $K[G]$ is semisimple.*

Furthermore, for all the remaining fields the semisimplicity question is equivalent to that of the rational group ring. We remark finally on a related ring theoretic problem. Let A be a finitely generated algebra over a field K. If A is commutative then it follows from the Nullstellensatz that JA is a nil ideal. The question here is whether the commutativity assumption can be eliminated. An affirmative answer proves the semisimplicity of $Q[G]$.

Now let char $K = p > 0$. Then of course we have the following analog of Theorem 5.1. This was proved by Passman (1962 [64]) with some additional assumptions on the field and then in general by Chalabi (1969 [12]) and Passman (1970 [68]).

THEOREM 5.2. *Let K be a field of characteristic $p > 0$ which is not an algebraic extension of $GF(p)$. If G is a p'-group then $K[G]$ is semisimple.*

It is natural to ask whether the converse to the above is true. The answer is emphatically no. There are many examples of groups G having elements of order p whose group rings are semisimple. For example we could take the Wreath products $G = Z \wr Z_p$ or $Z_p \wr Z$.

Suppose K is a fixed field of characteristic $p > 0$ and H is a normal subgroup of G. We say that H carries the radical of $K[G]$ if $JK[G] = JK[H] \cdot K[G]$. Observe that if H carries the radical then it controls it, but the converse is not true. Our goal in this subject is to find an appropriate carrier subgroup H of G such that the structure of $JK[H]$ is reasonably well understood. Of course the latter statement is somewhat vague but we would certainly insist that $JK[H]$ be so simple in nature that we can at least decide easily whether or not it is zero. Now if R is a ring with 1 we let $NR = \{\alpha \in R \mid \alpha R \text{ is nilpotent}\}$ be its nilpotent radical. Thus NR is the join of all the nilpotent ideals of R, so it is clearly a nil ideal but it need not be nilpotent. The following result shows that $\Delta^+(G)$ carries $NK[G]$ and satisfies our requirements on the simplicity of its structure.

THEOREM 5.3 (Passman 1970 [67]).

$$NK[G] = JK[\Delta^+(G)] \cdot K[G]$$
$$= \bigcup_W JK[W] \cdot K[G]$$

where W runs through all finite normal subgroups of G.

Furthermore $NK[G]$ is nilpotent if and only if $\Delta^+(G)$ contains only finitely many p-elements.

Unfortunately it is not true in general that $JK[G] = NK[G]$. However there is a good deal of evidence to support the premise that perhaps $JK[G] = NK[G]$, at least for G finitely generated. This then leads us to define a new radical for group rings as follows. If R is a ring with 1 we set

$$N^*R = \{\alpha \in R \mid \alpha S \text{ is nilpotent for all finitely generated subrings } S \subseteq R\}.$$

If $R = K[G]$ then it is immediate that $\alpha \in N^*K[G]$ if and only if $\alpha \in NK[H]$ for all finitely generated subgroups H of G containing the support of α. Therefore we can conclude easily that if $JK[H] = NK[H]$ for all finitely generated subgroups H of G, then $JK[G] = N^*K[G]$. While it is probably a little premature to do so, we nevertheless propose the following

CONJECTURE 5.4. $JK[G] = N^*K[G]$.

This is known to be true for G solvable, linear and trivially for G locally finite.

In order to study this N^* radical it is necessary to introduce some new group theoretic concepts. Let H be a subgroup of G. We say that H has locally finite index in G and write $[G:H] = \mathrm{lf}$ if $[L: L\cap H] < \infty$ for all finitely generated subgroups $L \subseteq G$. We now define

$$\Lambda = \Lambda(G) = \{x \in G \mid [G: C_G(x)] = \mathrm{lf}\}.$$

Then Λ is a subgroup of G and we let $\Lambda^+(G)$ be its torsion subgroup which is easily seen to exist.

LEMMA 5.5. *$\Lambda^+(G)$ is a characteristic locally finite subgroup of G. Furthermore, if $H \lhd G$ with $H \subseteq \Lambda^+(G)$, then $\Lambda^+(G/H) = \Lambda^+(G)/H$.*

This latter radical-like property of Λ^+, though simple to prove, comes as somewhat of a surprise since the analogous result for Δ^+ is not true. The following is an easy consequence of Theorem 5.3.

THEOREM 5.6 (Passman 1974 [79]).

$$N^*K[G] = JK[\Lambda^+(G)] \cdot K[G].$$

Thus $\Lambda^+(G)$ carries $N^*K[G]$. Note however that this subgroup does not as yet meet our criteria for simplicity of structure since $\Lambda^+(G)$ can be an arbitrary, locally finite group. We will discuss this problem further in the next section.

Let $\mathbf{O}_p(G)$ denote the maximal normal p-subgroup of G. Then we have

THEOREM 5.7 (Dyment-Zalesskii [19]). *Suppose that* $\mathbf{O}_p(\Delta(G)) = \langle 1 \rangle$. *Then*

$$N(K[G]/NK[G]) = 0.$$

COROLLARY 5.8 (Passman 1974 [79]).

$$N^*(K[G]/N^*K[G]) = 0.$$

Hence we see that N^* exhibits certain radical-like behavior at least in the case of group rings. Furthermore we have

THEOREM 5.9. (Passman 1974 [117]). *Suppose that* $JK[G] = N^*K[G]$ *and that* R *is a* K*-algebra with* $JR = 0$. *Then*

$$JR[G] = N^*R[G] = R \otimes_K N^*K[G].$$

Finally we remark on the relationship between these ideas for $K[G]$ and for rings in general. First it is not true for all rings R that $JR = N^*R$ since N^*R is a nil ideal. Moreover it is not even true that N^* is a radical for arbitrary rings, as we see from the following example of G. M. Bergman. Let $E = K[\varepsilon, \eta]$ be the free associative K-algebra in the two indeterminates ε and η and for each integer $n \geqq 0$ let I_n be the ideal $I_n = (E\,\varepsilon\,\eta^n\,\varepsilon\,E)^{n+1}$. If $I = \sum_n I_n$ then I is the ideal of E spanned by all monomials which for some integer n involve $n+1$ disjoint occurrences of $\varepsilon\eta^n\varepsilon$. Let $R = E/I$ so that R is a K-algebra generated by $\bar{\varepsilon}$ and $\bar{\eta}$ the images of ε and η. Now $R\,\bar{\varepsilon}\,\bar{\eta}^n\,\bar{\varepsilon}\,R$ is nilpotent, so clearly

$$\begin{aligned} NR &\supseteq \sum_{n=0}^{\infty} R\bar{\varepsilon}\bar{\eta}^n\bar{\varepsilon}R \\ &= R\bar{\varepsilon}R\bar{\varepsilon}R = (R\bar{\varepsilon}R)^2. \end{aligned}$$

In other words $R\bar{\varepsilon}R$ is nilpotent modulo NR. But $\bar{\varepsilon} \notin NR$ since otherwise $(R\bar{\varepsilon}R)^m = 0$ for some m and then $(\varepsilon\eta^m)^m \in I$, clearly a contradiction. Thus $N(R/N(R)) \neq 0$ and hence $N^*(R/N^*(R)) \neq 0$ since R is finitely generated.

6. Semisimplicity (special families)

The solution of the semisimplicity problem for group rings of solvable groups represents the high water mark of our knowledge. Again we consider only fields of characteristic $p > 0$ and we say that G is an fc group or a Δ-group if $G = \Delta(G)$, that is, if all conjugacy classes of G are finite. Suppose now that G has a normal Δ-subgroup H and an element $h \in H$ of order p with $[G\colon C_G(h)] =$ lf. Then it follows easily that $JK[G]$ is nonzero. Furthermore, at least in the case of solvable groups, the converse also holds. This was proved in a series of three major steps which we outline below.

Let $H \lhd G$ and let I be a nonzero ideal of $K[G]$. An intersection theorem is a result which guarantees that the intersection ideal $I \cap K[H]$ is nonzero. A number of such theorems occur in the literature under varying assumptions on H, G/H and I. For solvable groups G, Zalesskii (1973 [103]) proved an intersection theorem of tremendous strength. He showed that G has a characteristic Δ-subgroup G_0 such that for any nonzero ideal I of $K[G]$ we have $I \cap K[G_0] \neq 0$. If one thinks of the elements of $\Delta(G)$ as the almost central elements of G, then the group G_0 is very roughly the almost center of the almost Fitting subgroup of G. This, then, completes the first step of the proof. The normal Δ-subgroup of G has been found and we denote this Zalesskii subgroup of G by $G_0 = \mathfrak{Z}(G)$.

We mention now the following important result of Wallace (1967 [93]). Let $H \lhd G$ and let $\alpha \in JK[G] \cap K[H]$. If $x \in G$ has infinite order modulo H then, for some integer n,

$$\alpha\alpha^{x}\alpha^{x^2}\cdots\alpha^{x^n} = 0.$$

Of course for any integer $s \neq 0$, x^s also has infinite order modulo H and thus we obtain infinitely many such equations with varying $n = n(s)$, one for each s. The second step in the solvable group argument involves using these Wallace equations to find information about G one element at a time. To be more precise, suppose that $JK[G] \neq 0$, so that by the above we can fix a nonzero element

$$\alpha \in JK[G] \cap K[\mathfrak{Z}(G)].$$

We then define the p-quotient of α to be the set of p-elements of the form $g_1 g_2^{-1}$ with $g_1, g_2 \in \operatorname{supp} \alpha$. Thus p-quot α is a finite but possibly empty subset of $\mathfrak{Z}(G)$. The main result of (Hampton-Passman 1972 [37]) is that for any fixed $x \in G$ there exists $n > 0$ and $h \in p$-quot α with $x^n \in C_G(h)$. This is proved by a close study of the Wallace equations for x and α and uses the fact that $\mathfrak{Z}(G)$ is a solvable Δ-group.

It remains, finally, to somehow globalize this local information. If W is a subgroup of G, let us define its root set by $\sqrt{W} = \{x \in G \mid x^n \in W \text{ for some } n > 0\}$. Then by the above we have clearly

$$G = \bigcup_{p-\mathrm{Quot}\,\alpha} \sqrt{(C_G(h))}.$$

The problem now is to obtain something useful from this sort of equation. More generally, if G is a group and if $W_1, W_2, \cdots, W_m$ are finitely many subgroups, what can we deduce from $G = \bigcup \sqrt{W_i}$? We would like to show that for some i, $[G: W_i] = \mathrm{lf}$, but this cannot always be so since the special case of $m = 1$ and $W_1 = \langle 1 \rangle$ is clearly equivalent to the general Burnside problem. Fortunately, however, it is true in the case of solvable groups, as is shown in (Passman 1973 [75]) and thus there exists an element $h \in p$-quot α with $[G: C_G(h)] = \mathrm{lf}$. This completes the third step in the argument and yields

THEOREM 6.1 (Hampton-Passman-Zalesskii). *Let K be a field of characteristic $p > 0$ and let G be a solvable group. Then $JK[G] \neq 0$ if and only if $\mathfrak{Z}(G)$ has an element h of order p with $[G: C_G(h)] = \mathrm{lf}$.*

It is amusing to note that, chronologically, these steps were proved in precisely the reverse order. Somewhat later Zalesskii [104] generalized his intersection theorem to prove that if G is solvable and finitely generated then $JK[G] = NK[G]$. From this we conclude further that $JK[G] = N^*K[G]$ for G solvable, and hence by Theorem 5.6 the structure of the radical $JK[G]$ is determined by that of the solvable locally finite subgroup $\Lambda^+(G)$.

Let us now turn to another family of groups, the linear groups. It is apparent that here there are actually three distinct problems, namely linear groups in characteristic 0, characteristic p and finite characteristic $q \neq p$. Suppose first that G is a linear group in a characteristic different from p. If H is a finitely generated subgroup of G then it follows easily that H has a subgroup H_0 of finite index which is residually a finite p'-group. This then implies that $K[H_0]$ is semi-simple and hence $JK[H]$ is nilpotent, so certainly $JK[H] = NK[H]$. Now from this we conclude that $JK[G] = N^*K[G]$ and thus the structure of this radical can be obtained by studying the locally finite linear group $\Lambda^+(G)$. On the other hand, results on linear groups in characteristic p require more difficult proofs.

Consider the following clever argument of Zalesskii (1971 [101]). Let G be a linear group in characteristic p so that $G \subseteq L_u$, the ring of $u \times u$ matrices over the field L. Suppose for simplicity that G is irreducible, that is, that G spans L_u, and suppose also that $JK[G]$ is a nil ideal. If $\alpha = \sum_1^m a_i g_i$ is a fixed nonzero element of $JK[G]$ and if $x \in G$, then αx is nilpotent, so by Lemma 2.1 we have for some $n > 0$,

$$0 = (\alpha x)^{p^n} = \sum_1^m a_i^{p^n} (g_i x)^{p^n} + \beta$$

with $\beta \in [K[G], K[G]]$. Thus we conclude from the nature of β that for some $i \neq j$, $(g_i x)^{p^n}$ and $(g_j x)^{p^n}$ are conjugate in G. Hence if s denotes the trace map in L_u then, since char $L = p$, we have

$$(s(g_i x))^{p^n} = s((g_i x)^{p^n}) = s((g_j x)^{p^n}) = (s(g_j(x))^{p^n}$$

and therefore $s((g_i - g_j)x) = 0$. Now for each $i \neq j$ define

$$S_{ij} = \{\gamma \in L_u \,|\, s((g_i - g_j)\gamma) = 0\}.$$

Then S_{ij} is a hypersurface in L_u and by the above we have $G = \bigcup_{i \neq j}(S_{ij} \cap G)$ which is a rather stringent condition on an irreducible linear group. From this it is not hard to deduce more generally that if $O_p(G) = \langle 1 \rangle$ then $JK[G]$ is nilpotent. Somewhat later in (Passman 1973 [77], [78]) these groups were studied without the no normal p-subgroups assumption by a blending of the above argument of Zalesskii and the earlier work on solvable groups. The answer was essentially the same.

THEOREM 6.2 (Zalesskii 1971 [101], Passman 1973 [77], [78]). *Let K be a field of characteristic $p > 0$ and let G be a linear group in characteristic p. Then $JK[G] \neq 0$ if and only if $\mathscr{L}(G)$, a certain normal Δ-subgroup of G, has an element h of order p with $[G: C_G(h)] = $ lf.*

Furthermore from this it follows easily that here too we have $JK[G] = N^*K[G]$.

Thus we have seen that for at least two large families of groups, namely the solvable and linear groups, we always have $JK[G] = N^*K[G]$. Moreover, this clearly also holds for G locally finite. Therefore, by Theorem 5.6, a complete description of $JK[G]$ will follow from a consideration of the locally finite subgroup $\Lambda^+(G)$. It is clear then that the study of locally finite groups is now very important and we discuss what little is known here in the remainder of this section.

For G locally finite, we let G^p denote the characteristic subgroup generated by all p-elements of G and we set $\Delta^p = (\Delta^+)^p$ and $\Lambda^p = (\Lambda^+)^p$. We then have the following elementary

LEMMA 6.3. *Let G be locally finite and let* char $K = p$.

i. $JK[G] = JK[G^p] \cdot K[G]$.

ii. *If $P = O_p(G)$, then the kernel $\omega(K[P]) \cdot K[G]$ of the natural map $K[G] \to K[G/P]$ is contained in the radical of $K[G]$ and hence*

$$JK[G]/(\omega(K[P]) \cdot K[G]) = JK[G/P].$$

As a consequence of (i) we see that if H carries $JK[G]$ then so does H^p. Furthermore, by (ii), we can usually assume that $O_p(G) = \langle 1 \rangle$. The appropriate results for solvable and linear groups are as follows.

THEOREM 6.4 (Zalesskii [104], Passman 1974 [79]). *Let G be a locally finite group and let* char $K = p$. *If $\bar{G} = G/O_p(G)$ then*

i. *for G solvable*

$$JK[\bar{G}] = JK[\mathfrak{Z}^p(\bar{G})] \cdot K[\bar{G}];$$

ii. *for G linear in characteristic* 0 *or p*

$$JK[\bar{G}] = JK[\Delta^p(\bar{G})] \cdot K[\bar{G}].$$

Thus the structure of $JK[G]$ for solvable groups and most linear groups is completely determined. Only the case of locally finite linear groups in characteristic $q \neq p$ remains open. We note that in the above two examples $JK[\bar{G}]$ is carried by a normal Δ-subgroup of $\bar{G}$.

Now suppose that H is a normal Δ-subgroup of G. Then any finite subset of H is contained in some finite normal subgroup N of H and clearly N is subnormal in G. Thus perhaps the finite subnormal subgroups of G play a role in the semisimplicity problem here. However, since subnormality in infinite groups is not usually a well-behaved condition and since containment in $JK[G]$ is really a local property, it makes more sense to consider local subnormality instead. Let A be a finite subgroup of G. We say that A is locally subnormal in G if $A \lhd\lhd H$ for all finite subgroups H containing A. If $\int(G)$ is the subgroup of G generated by all such locally subnormal subgroups then we have

LEMMA 6.5. *Suppose $H \lhd G$ and $JK[G] = JK[H] \cdot K[G]$. Then $H \supseteq \int^p(G)$. Furthermore, if $H = \int(H)$ then $H^p = \int^p(G)$ and $JK[G] = JK[\int^p(G)] \cdot K[G]$.*

It then follows from Theorem 6.4 that $JK[\bar{G}]$ is carried by $\int^p(\bar{G})$ in the appropriate two cases and, furthermore, that in these cases $\int^p(\bar{G})$ is a Δ-group.

It is quite possible that $\int^p(G)$ always carries the radical in case G is locally finite with $O_p(G) = \langle 1 \rangle$, but we do not know for sure. Nevertheless, since $\int^p(G)$ is at least always contained in any carrier subgroup, it is certainly worthwhile to study the structure of this characteristic subgroup. Now it turns out that $\int^p(G)$ need not always be a Δ-group even if $O_p(G) = \langle 1 \rangle$ but we do have the following interesting result. For each integer $n \geqq 1$ let $\int_n(G)$ be the subgroup of G generated by all the locally subnormal subgroups of composition length at most n. Then we have

THEOREM 6.6 (Passman 1974 [80]). *Let G be a locally finite group with $O_p(G) = \langle 1 \rangle$. Then*

$$\int^p(G) = \bigcup_{n=1}^{\infty} \int_n^p(G)$$

is the ascending union of the characteristic Δ-subgroups $\int_n^p(G)$.

Moreover from a group ring point of view we have at least

Theorem 6.7. (Passman [119]). *Let G be a locally finite group with $O_p(G) = \langle 1 \rangle$ and let* $\operatorname{char} K = p$. *Then $I = JK[\int^p(G)] \cdot K[G]$ is a radical ideal of $K[G]$. In addition if $\Delta(G) = \langle 1 \rangle$ then I is a prime ideal.*

Finally we remark that Formanek (1972 [25]) has proved the semisimplicity of the group ring of the infinite symmetric group and (Passman [120]) studies locally solvable groups and linear groups.

7. Primitivity

We now come to a problem somewhat related to semisimplicity, namely primitivity. Here, instead of asking whether $K[G]$ has enough irreducible representations, we in fact consider whether perhaps one could be enough. Now a primitive ring is always prime and hence, by Theorem 3.4, a necessary condition for $K[G]$ to be primitive is that $\Delta^+(G) = \langle 1 \rangle$. For the longest time no examples of such group rings were known with the exception, of course, of $G = \langle 1 \rangle$ and it was thought that possibly $K[G]$ could not otherwise be primitive. However this illusion was shattered by the following.

THEOREM 7.1 (Formanek-Snider 1972 [31]). *Let G be a locally finite, countable group. Then $K[G]$ is primitive if and only if $JK[G] = 0$ and $\Delta(G) = \langle 1 \rangle$.*

In particular, the group ring of the infinite symmetric group is primitive for all fields. The neatest proof of this theorem now appears in Fisher-Snider [23].

In addition, Formanek and Snider showed that for any group ring $K[G]$ one can find an over group $H \supseteq G$ with $K[H]$ primitive. While this now follows from a later result of Formanek on free products, we nevertheless sketch its original proof here. Define a sequence of groups $\{G_i\}$ and a sequence of $K[G_i]$-modules $\{V_i\}$ inductively by $G_1 = G$, $V_1 = K[G_1]$ and

$$G_{n+1} = GL(V_n),\ V_{n+1} = K[G_{n+1}] \oplus V_n.$$

Then $G_n \subset G_{n+1}$ and $V_n \subset V_{n+1}$ so we can let $H = \bigcup G_i$ and $V = \bigcup V_i$. Clearly H is a group containing G and V is a $K[H]$-module. Now $K[G_{n+1}]$ acts irreducibly on V_n, so we see that $K[H]$ acts irreducibly on V. Moreover, $K[G_{n+1}]$ acts faithfully on V_{n+1}, so $K[H]$ acts faithfully on V. Thus $K[H]$ is primitive with V as a faithful irreducible module.

One of the surprising aspects of this subject is that the primitivity of $K[G]$ depends on the size of the field and not only on its characteristic. Suppose F is a field extension of K and I is a nonzero ideal of $F[G]$. If $\{f_i\}$ is a basis for F/K then every element of $F[G]$ can be written as $\Sigma\alpha_i f_i$ with $\alpha_i \in K[G]$. Let $\alpha \in I$, $\alpha \neq 0$ be so chosen that in $\alpha = \Sigma_1^n \alpha_i f_i$ we have n minimal. If $x \in G$ then

$$\alpha_1 x\alpha - \alpha x\alpha_1 = \sum_2^n (\alpha_1 x\alpha_i - \alpha_i x\alpha_1) f_i \in I$$

so by the minimality of n we conclude that $\alpha_1 x\alpha_i = \alpha_i x\alpha_1$ for all x. Furthermore, by studying these linear identities under the additional assumption that $\Delta(G) = \langle 1 \rangle$, we can then obtain

THEOREM 7.2 (M. Smith 1971 [87]). *Let $F \supseteq K$ and let I be a nonzero ideal in $F[G]$. If $\Delta(G) = \langle 1 \rangle$ then $I \cap K[G] \neq 0$.*

COROLLARY 7.3 (Passman 1973 [76]). *Let $F \supseteq K$ and let $\Delta(G) = \langle 1 \rangle$. If $K[G]$ is primitive then so is $F[G]$.*

The assumption $\Delta(G) = \langle 1 \rangle$ in the above corollary is really unexpected. More reasonable perhaps would be the condition associated with primeness, namely $\Delta^+(G) = \langle 1 \rangle$. However this stronger assumption is indeed necessary in view of the following.

THEOREM 7.4 (Passman 1973 [76]). *Suppose $K[G]$ is a primitive group ring and that the cardinality of K is larger than the cardinality of G. Then $\Delta(G) = \langle 1 \rangle$.*

There still remained the question of whether $K[G]$ could ever be primitive if $\Delta(G) \neq \langle 1 \rangle$. This was settled in the affirmative by Formanek as follows. Let A

and B be nonidentity groups with, say, A infinite and let R be a ring without zero divisors. Suppose further that the cardinality of R is at most equal to the cardinality of $G = A * B$, where the latter is the free product of the two groups. If A is the larger of the two groups then clearly $R[G]$ and A have the same cardinality and hence there exists a one-to-one correspondence $A \to R[G]$ given by $a \to \alpha(a)$ and $1 \to 0$. It can then be shown that there exist group elements x, y, z, w depending on the nature of $\alpha(a)$ such that if $\beta(a)$ is defined by

$$\beta(a) = 1 + x\alpha(a)y + z\alpha(a)w$$

then the right ideal $\sum_{a \neq 1} \beta(a)R[G]$ is proper. Therefore we can extend this ideal to a maximal right ideal M of $R[G]$ and it then follows easily that $R[G]$ acts faithfully on the irreducible module $R[G]/M$. Furthermore, free products of finite groups can also be handled to yield

THEOREM 7.5 (Formanek 1973 [28]). *Let A and B be nonidentity groups not both of order 2 and let R be a ring without zero divisors. If $G = A * B$ and $|G| \geqq |R|$ then $R[G]$ is primitive.*

Moreover, since $\Delta(A * B) = \langle 1 \rangle$ unless $A = B = Z_2$, the above and Corollary 7.3 yield

COROLLARY 7.6 (Formanek 1973 [28]). *Let A and B be nonidentity groups not both of order 2. If $G = A * B$ then $K[G]$ is primitive.*

Now let C be a torsion-free abelian group. Then $R = K[C]$ has no zero divisors and $R[G] = K[C \times G]$. Thus, for example, with $A = B = C = Z$, the infinite cyclic group, we have by the above and Theorem 7.4

COROLLARY 7.7 (Formanek 1973 [28]). *Let $G = Z \times (Z * Z)$. Then $K[G]$ is primitive if and only if K is countable.*

We observe in the above example that $K[G]$ is primitive if and only if K is small. On the other hand, as is indicated in (Passman 1973 [76]), there are results which go in precisely the reverse direction. To be more concrete, we first state the following beautiful

THEOREM 7.8 (Bergman 1971 [5]). *Let A be a finitely generated, free abelian group and let G act on A in such a way that it and all its subgroups of finite index act rationally irreducibly. If I is a G-invariant ideal of $K[A]$ then either $I = 0$ or $K[A]/I$ is finite dimensional.*

Using this we can then easily show

COROLLARY 7.9. *Let $A \neq \langle 1 \rangle$ be a finitely generated, free abelian group of* *rank at least* 2 *and let H act faithfully on A in such a way that it and all its subgroups of finite index act rationally irreducibly. If $G = A \otimes_\sigma H$ and if $K^\bullet$ has an element of infinite order then $K[G]$ is primitive.*

On the other hand, if H in the above is polycylic and if K is algebraic over a finite field then $K[G]$ is never primitive by a deep result of Roseblade (Th. 9.9) which will be discussed later.

8. Augmentation annihilators

There is an elementary lemma on annihilators of elements in the augmentation ideal which has a number of lovely applications. For example, it can be used to determine when $K[G]$ is regular and when certain $K[G]$-modules are injective. The lemma is as follows.

LEMMA 8.1. *Let $\{g_\nu\}$ be a subset of G. Then the elements $1 - g_\nu$ have a common nonzero right or left annihilator if and only if the subgroup $\langle g_\nu \mid all\ \nu \rangle$ is finite.*

To see this, suppose first that there exists $\alpha \neq 0$ with $(1 - g_\nu)\alpha = 0$ for all ν. Then $\alpha = g_\nu \alpha$ and hence clearly $\alpha = h\alpha$ for all $h \in H = \langle g_\nu \mid \text{all } \nu \rangle$. Thus H permutes the finitely many elements of supp α by left multiplication, and since H acts faithfully in this manner we conclude that $|H| < \infty$. Conversely if H is finite then each $1 - g_\nu$ annihilates $\alpha = \sum_{h \in H} h$.

As an application, suppose that $K[G]$ is von Neumann regular. Now in a regular ring any finitely generated proper right ideal is generated by an idempotent and hence the right ideal $\sum_1^n (1 - g_i)K[G]$ has a nonzero left annihilator. Thus, by the above, G is locally finite and with a little additional work we obtain

THEOREM 8.2 (Villamayor 1959 [92], Connell 1963 [15]). *The group ring $K[G]$ is von Neumann regular if and only if G is locally finite and has no elements of order p in case K has characteristic $p > 0$.*

Now let us consider injective modules. To start with, we have the following elementary argument. Let V be an injective right R-module and let $\alpha_1, \alpha_2, \cdots, \alpha_n \in R$ have no nonzero common right annihilator in R. Then the map

$$\beta \to (\alpha_1\beta, \alpha_2\beta, \cdots, \alpha_n\beta)$$

is an R-module injection of R into R^n. Moreover, if v is any fixed element of V

then, since V is injective, the map $R \to V$ given by $\beta \to v\beta$ extends to a map $\theta: R^n \to V$. Hence if $\delta_i = (0, 0, \cdots, 1, \cdots, 0) \in R^n$ then

$$v = \theta(1) = \theta(\Sigma \delta_i \alpha_i) = \Sigma \theta(\delta_i)\alpha_i \in \Sigma V\alpha_i$$

and we have shown

LEMMA 8.3. *Let V be an injective right R-module. If $\alpha_1, \alpha_2, \cdots, \alpha_n \in R$ then either they have a common right annihilator in R or $V = V\alpha_1 + V\alpha_2 + \cdots + V\alpha_n$.*

Furthermore, by combining this with Lemma 8.1 we have

LEMMA 8.4. *Let V be an injective $K[G]$-module. If $V \neq V\omega(K[G])$ then G is locally finite.*

This of course applies beautifully to self-injective group rings and indeed Connell (1963 [15]) made the following two observations. First, if G is finite then $K[G]$ is always self-injective; second, if $K[G]$ is self-injective then, by the above, G is at least always locally finite. To complete the argument it was necessary to show in the latter that G cannot be a countably infinite locally finite group and this was finally done in the paper of Renault (1971 [82]). Here certain deep group theoretic results were used to reduce the problem to p^∞-groups where a key lemma, attributed to Pascaud, could be applied. It should be noted however that a minor modification of this lemma actually proves the theorem directly. Moreover, at exactly the same time, Malliavin (1971 [59]) proved a stronger result which also implies this lemma. Thus we have

THEOREM 8.5 (Connell-Malliavin-Pascaud-Renault). *The group ring $K[G]$ is self-injective if and only if G is finite.*

A slightly simpler proof has recently been given by Farkas (1973 [20]). Furthermore we remark that the first proof of the above in characteristic 0 was given by Gentile (1969 [107]).

We now consider the possibility of other modules being injective. We first require

LEMMA 8.6. *Let G be a locally finite group with no elements of order p if* char $K = p > 0$. *Let V be a $K[G]$-module which satisfies the descending chain condition on annihilators. Then V is injective.*

PROOF. Let I be a right ideal of $K[G]$ and let $\theta: I \to V$ be a $K[G]$-module homomorphism. Then, by the descending chain condition on annihilators, there exists a finite subgroup H_0 of G such that the left annihilator in V, $l(I \cap K[H_0])$' is minimal over all finite subgroups of G. Since all $K[H_0]$-modules are injective we can extend $\theta: I \cap K[H_0] \to V$ to a map of $K[H_0]$ to V by sending $1 \to v_0$. We claim that $\phi: K[G] \to V$ defined by $\alpha \to v_0\alpha$ extends θ.

Fix some finite subgroup $H \supseteq H_0$. Since all $K[H]$-modules are injective, $\theta: I \cap K[H] \to V$ can be extended to a map from $K[H]$ to V defined by $1 \to v$. Thus for all $\beta \in I \cap K[H_0]$ we have $\theta(\beta) = \theta(1 \cdot \beta) = v\beta$ and also $\theta(\beta) = v_0\beta$, so we conclude that $v - v_0 \in l(I \cap K[H_0]) = l(I \cap K[H])$, by the minimality of $l(I \cap K[H_0])$. Hence the map $K[H] \to V$ given by $1 \to v_0$ also extends θ on $I \cap K[H]$ and the lemma is proved.

If $V_0 = K[G]/\omega(K[G])$ is the principal module of $K[G]$ then Lemmas 8.4 and 8.6 essentially yield

THEOREM 8.7 (Farkas-Snider [22]). *The principal module V_0 is an injective $K[G]$-module if and only if G is locally finite with no elements of order p in case* char $K = p > 0$.

In addition Lemma 8.6 yields the easy direction of the following.

THEOREM 8.8 (Farkas-Snider [22]). *Let G be a countable locally finite group with no elements of order p in case* char $K = p > 0$. *Then all irreducible $K[G]$-modules are injective if and only if they are all finite dimensional over their commuting rings.*

Furthermore, if either char $K = p > 0$ or char $K = 0$ and K contains all roots of unity, then the above is also equivalent to G having an abelian subgroup of finite index. See (Goursaud-Valette [109]) for a different proof with somewhat simpler ring theoretic arguments. Finally it is quite easy to see that V_0 is projective if and only if G is a finite group of order prime to the characteristic of K.

Let us now consider another question of interest. Suppose I is a commutative ideal in a ring R. If $\alpha, \beta \in I$ and $\gamma \in R$ then $(\alpha\gamma)\beta = \beta(\alpha\gamma)$ and $\alpha(\gamma\beta) = (\gamma\beta)\alpha$ so γ commutes with $\beta\alpha$ and hence I^2 is a central ideal. Now let I be a nonzero central ideal of $K[G]$. If $\alpha \in I$, $\alpha \neq 0$ and if $x, y \in G$ then y commutes with α and αx so

$$\alpha x = (\alpha x)^y = \alpha^y x^y = \alpha x^y$$

and hence $\alpha(1 - x^y x^{-1}) = 0$. Thus Lemma 8.1 yields

THEOREM 8.9. (Wallace 1969 [94]). *$K[G]$ has a nonzero central ideal if and* *only if G' is finite. Furthermore if I is a commutative ideal of $K[G]$ then I^2 is central.*

Wallace (1969 [94]) then goes on to study the possibility that $JK[G]$ is commutative or central and reasonably tight characterizations are obtained.

9. Chain conditions

Let us now consider Artinian and Noetherian group rings. Suppose A and B are subgroups of G with A properly containing B. Then it is quite easy to see that $\omega(K[A]) \cdot K[G]$ properly contains $\omega(K[B]) \cdot K[G]$ and from this we conclude that G inherits the chain conditions of $K[G]$. The first result of interest here is the beautiful

THEOREM 9.1 (Connell 1963 [15]). *$K[G]$ is Artinian if and only if G is finite.*

Furthermore this has been generalized somewhat to yield

THEOREM 9.2 (Woods 1971 [96], Renault 1971 [82]). *$K[G]$ is perfect if and only if G is finite.*

A third proof of the above appears in Passman's book (1971 [69]).

On the other hand, little progress has been made in determining when group rings are semiperfect or semilocal. In fact it is clear that results here are intimately related to the semisimplicity problem and will therefore be very difficult to come by. The only general fact of interest, due to Woods (1974 [97]), is that either of the above conditions implies that G is a torsion group. Some other results can be found in (Valette 1972 [122]). We do however also have the following (see (Goursaud 1973 [108]) for an alternate proof.)

THEOREM 9.3. *$K[G]/N^*K[G]$ is Artinian if and only if* char $K = 0$ *and G is finite or* char $K = p > 0$ *and G is locally finite with $[G:O_p(G)] < \infty$.*

PROOF. Since $N^*K[G]$ is a nil ideal, the characteristic zero result follows from Corollary 2.3 and Theorem 9.1. Thus let char $K = p > 0$ and suppose that $K[G]/N^*K[G]$ is Artinian. Since $N^*K[G] = JK[\Lambda^+(G)] \cdot K[G]$ and

$$\omega(K[\Lambda^+(G)]) \supseteq JK[\Lambda^+(G)]$$

it follows that $K[G/\Lambda^+(G)]$ is a homomorphic image of the Artinian ring $K[G]/N^*K[G]$. Hence by Theorem 9.1, $G/\Lambda^+(G)$ is finite and this therefore implies that $G = \Lambda^+(G)$ is locally finite. We may now clearly assume that $O_p(G) = \langle 1 \rangle$.

By assumption, $K[G]/JK[G]$ is a finite direct sum of full matrix rings over division rings which are certainly algebraic over K. Hence by an argument of

Farkas and Snider [22], since char $K = p > 0$, these division rings are in fact fields. Moreover, since $O_p(G) = \langle 1 \rangle$, G is mapped isomorphically into $K[G]/JK[G]$ and hence G is a linear group in characteristic p with $O_p(G) = \langle 1 \rangle$. Therefore by a theorem of Zalesskii (1971 [101]) (see Theorem 6.2 and the comments preceding it) $JK[G]$ is nilpotent. Thus we see that $K[G]$ is perfect and Theorem 9.2 yields the result.

As an example of the difficulty of these problems, let us observe that $K[G]$ is semiperfect when $JK[G] = \omega(K[G])$ and that we know almost nothing about when this latter equality can occur. Suppose that $JK[G] = \omega(K[G])$ and char $K = p > 0$. Then it is quite easy to see at least that G is a p-group. Let us suppose further that $G \neq \langle 1 \rangle$ is finitely generated. Since $\omega(K[G])$ must then be a finitely generated right ideal of $K[G]$, Nakayama's lemma yields

$$\omega(K[G])^2 = \omega(K[G]) \cdot JK[G] < \omega(K(G]).$$

Now if "$^-$" denotes the homomorphism $^-: K[G] \to K[G]/\omega(K[G])^2$ and if $x, y \in G$ then

$$xy - yx = (1 - x)(1 - y) - (1 - y)(1 - x) \in \omega(K[G])^2$$

so $\bar{x}\bar{y} = \bar{y}\bar{x}$. Thus $\bar{G}$, the image of G, is a finitely generated abelian p-group and hence $\bar{G}$ is finite. Furthermore $\bar{G} \neq \langle 1 \rangle$ since otherwise $\overline{x - 1} = 0$ for all $x \in G$ and $\omega(K[G])$ would be in the kernel. Finally with a little more work we obtain the only non-trivial result on this equality, namely

THEOREM 9.4 (Lihtman 1963 [58]). *Let K be a field of characteristic $p > 0$ and let G be a finitely generated group. If $JK[G] = \omega(K[G])$ then either G is a finite p-group or G has as a homomorphic image an infinite residually finite p-group.*

In addition, if G has period p, then it follows from Kostrikin's solution (1959 [54]) of the restricted Burnside problem that the above second possibility cannot occur. Hence at least in this special case G must necessarily be a finite p-group.

The question of when group rings are (right) Noetherian is also still open. We know of course that if $K[G]$ is Noetherian then the lattice of subgroups of G must satisfy the ascending chain condition. Conversely we have the following sufficient condition due to P. Hall (1954 [34]).

LEMMA 9.5. *Suppose G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each quotient G_i/G_{i-1} being either finite or cyclic. Then $K[G]$ is Noetherian.

If all of the above quotients are cyclic, then G is of course polycyclic and $K[G]$ offers an interesting noncommutative analog of polynomial rings. These group rings have been extensively studied and we briefly mention a few results below.

We note that a finitely generated nilpotent group is always polycyclic. To start with, we have the following analogs of the Nullstellensatz.

THEOREM 9.6 P. Hall 1959 [36]). *Let G be a finitely generated nilpotent group and let I be a primitive ideal in $K[G]$. Then the image of the center of G in $K[G]/I$ is algebraic over K.*

THEOREM 9.7 (Zalesskii 1971 [100]). *Let G be a finitely generated nilpotent group and let I be an ideal of $K[G]$. Then I is a primitive ideal if and only if it is maximal. Furthermore if I is prime and if G is mapped faithfully into $K[G]/I$ then $I = (I \cap K[\Delta(G)]) \cdot K[G]$.*

The latter, of course, says that $\Delta(G)$ controls I. Moreover, if G is also torsion free then $\Delta(G)$ is the center Z of G and we conclude that the map $I \to I \cap K[Z]$ induces a one-to-one correspondence between the prime ideals I of $K[G]$ with G faithfully embedded in $K[G]/I$ and the prime ideals L of $K[Z]$ with Z faithfully embedded in $K[Z]/L$. Finally if I is primitive here then $K[G]/I$ is isomorphic to a twisted group ring of G/Z over the field $F = K[Z]/(I \cap K[Z])$. Additional results can be found in Zalesskii (1965 [98], 1966 [99]).

We now consider groups which are not necessarily nilpotent. Let us say that K is an absolute field if K is an algebraic extension of a finite field. We then have

THEOREM 9.8 (P. Hall 1959 [36]). *Let G be a polycyclic group and suppose that K is not an absolute field. Then all irreducible $K[G]$-modules are finite dimensional over K if and only if G has an abelian subgroup of finite index.*

The situation with K an absolute field is different and difficult. The work here was begun by P. Hall (1959 [36]) who used Theorem 9.6 to handle the nilpotent case and the problem has just recently been completely solved as follows.

THEOREM 9.9 (Roseblade 1973 [85]). *Let G be a polycylic group and let K be an absolute field. Then all irreducible $K[G]$-modules are finite dimensional over K.*

Another result of interest is

THEOREM 9.10 (P. Smith 1972 [91]). *Let G be a polycyclic group. Then* *the Krull dimension of $K[G]$ is equal to the number of infinite cyclic factors in a defining series for G.*

Finally P. Smith (1970 [89]) considers the nature of ideals of the form $\bigcap_{n=1}^{\infty} I^n$ in $K[G]$ for G finitely generated and nilpotent.

10. Zero divisors

We now come to the famous zero divisor problem and other related questions. To start with, we say that $\alpha \in K[G]$ is a trivial unit if $\alpha = kx$ for some $k \in K$, $k \neq 0$ and $x \in G$. Of course all other units are then nontrivial. Now suppose that G has some elements of finite order and let H be a nonidentity finite subgroup. If $\alpha = \sum_{x \in H} x \in K[G]$ and if $h \in H$, then clearly $h\alpha = \alpha$. Thus $\alpha^2 = n\alpha$ with $n = |H|$, and from $\alpha(\alpha - n) = 0$ we see that $K[G]$ has proper divisors of zero. Furthermore, if $|K| > 3$ we can choose $r \in K$ with $r \neq 0, 1, 1/n$. Then $1 - r\alpha$ is a nontrivial unit in $K[G]$ with inverse $1 - s\alpha$ where $s = r(rn-1)^{-1}$. Finally, with a little additional work we obtain

THEOREM 10.1. *Let G be a group which is not torsion free. Then $K[G]$ has proper divisors of zero. Moreover, with the exception of $K = GF(2)$, $|G| = 2$ or 3 and $K = GF(3)$, $|G| = 2$ the group ring has nontrivial units.*

The exceptions above are just too small to have room for nontrivial units. Now what is really interesting here is the converse. Indeed, with, frankly, very little supporting evidence, it has been conjectured that if G is torsion free then $K[G]$ has no zero divisors or nontrivial units.

Most of the early work on this problem was group theoretic in nature. We say that G is a up-group (unique product group) if given any two finite nonempty subsets A and B of G, then there exists at least one element $x \in G$ which has a unique representation of the form $x = ab$ with $a \in A$, $b \in B$. Furthermore G is a tup-group (two unique products group) if given any two nonempty finite subsets A and B of G with $|A| + |B| > 2$, then there exist as least two uniquely represented elements in AB. Observe that if $\alpha, \beta \in K[G]$ and if $x = ab$ is a unique product in (supp α) (supp β), then x occurs in supp $\alpha\beta$. In fact, the coefficient of x is just the product of the coefficient of a in α and of b in β. This then yields

LEMMA 10.2. *If G is a* up-*group, then $K[G]$ has no proper divisors of zero. If G is a* tup-*group, then $K[G]$ has only trivial units.*

Now the tup-groups form a fairly large family of torsion-free groups. They include, for example, ordered and left-ordered groups (Rudin-Schneider 1964 [86]), locally indicable groups (Higman 1940 [42]) and free groups. Furthermore, if $H \lhd G$ and both H and G/H are up-groups or tup-groups then so is G. Thus we have

THEOREM 10.3 (Bovdi 1960 [9]). *Let G be a group with a subnormal series $\langle 1\rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that each quotient G_i/G_{i-1} is torsion-free abelian. Then $K[G]$ has no proper divisors of zero and only trivial units.*

It is apparent that the existence of only trivial units is a more stringent condition than is the lack of proper zero divisors. Indeed we have the following argument. First, let R be a prime ring with nonzero elements a, b satisfying $ba = 0$. Then $aRb \neq 0$ but $(aRb)^2 = 0$ so there exists a nonzero $\alpha \in R$ with $\alpha^2 = 0$. Now if G is a torsion-free group then $K[G]$ is prime by Theorem 3.4 and hence, by the above, $K[G]$ has nontrivial zero divisors if and only if it has nontrivial elements of square zero. Moreover, if $\alpha^2 = 0$, $\alpha \neq 0$ then $(1+\alpha)(1-\alpha) = 1$ so $1-\alpha$ is a unit. In addition, $1-\alpha$ cannot possibly be trivial since if $1-\alpha = kg$ then $\alpha = 1 - kg \in K[\langle g\rangle]$ is nilpotent, a contradiction. Thus we have obtained the following well-known

THEOREM 10.4. *Let G be a torsion-free group. Then $K[G]$ has a proper zero divisor if and only if it has a nonzero element of square zero. Furthermore, if this occurs then $K[G]$ has a nontrivial unit.*

On the other hand, we have the following generalization of an unpublished result of S. K. Sehgal on integral group rings.

THEOREM 10.5. *Let G be a* up-*group and let* char $K = 0$. *Then $K[G]$ has only trivial units.*

PROOF. We may clearly assume that K is the field of complex numbers. For $\alpha = \sum a_x x$, $\beta = \sum b_x x$ in $K[G]$ we define $\bar{\alpha} = \sum \bar{a}_x x^{-1}$ and

$$(\alpha, \beta) = \sum \bar{a}_x b_x = \operatorname{tr} \bar{\alpha}\beta$$

where $\bar{a}$ denotes the complex conjugate of $a \in K$. Then $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ and (α, β) is a Hermitian inner product. As usual set $\|\alpha\| = (\alpha, \alpha)^{\frac{1}{2}}$.

Now let α be a unit in $K[G]$ with inverse β. Then $\alpha\beta = \beta\alpha = 1$ so $\bar{\alpha}\bar{\beta} = \bar{\beta}\bar{\alpha} = 1$ and hence $(\bar{\alpha}\alpha)(\beta\bar{\beta}) = 1$. Consider the unique product element gh in (supp $\bar{\alpha}\alpha$) (supp $\beta\bar{\beta}$). Since this element occurs in the group ring product, we have $1 = gh$ so $h = g^{-1}$. Furthermore, g^{-1} also belongs to supp $\bar{\alpha}\alpha$ since $\bar{\alpha}\alpha$ is $^{-}$-invariant and

$h^{-1} = g \in \mathrm{supp}\, \bar{\beta}\beta$. Since $g^{-1}g$ is also equal to 1, the uniqueness of this product yields $g = g^{-1}$ and hence $g = 1$ in this torsion-free group. Thus, by comparing coefficients in the product $(\bar{\alpha}\alpha)(\bar{\beta}\beta) = 1$, we have $(\mathrm{tr}\, \bar{\alpha}\alpha)(\mathrm{tr}\, \bar{\beta}\beta) = 1$ or $\| \alpha \| \cdot \| \beta \| = 1$. On the other hand, $\bar{\alpha}\beta = 1$ yields $(\alpha, \beta) = 1$ so $|(\alpha, \beta)| = \| \alpha \| \cdot \| \beta \|$. Hence by the Cauchy-Schwarz inequality, β is a scalar multiple of α. Say $\beta = k\alpha$ with $k \in K$.

Now $1 = \alpha\bar{\beta} = \bar{k}(\alpha\bar{\alpha})$ and we consider the unique product element in (supp α) (supp $\bar{\alpha}$). Certainly this element must be 1. On the other hand, if $g \in \mathrm{supp}\, \alpha$ then $g^{-1} \in \mathrm{supp}\, \bar{\alpha}$ and $1 = gg^{-1}$. Thus in order for 1 to be uniquely represented we can only have one group element in supp α and therefore α is a trivial unit.

Now recent work in this field has centered upon groups which are not left orderable. Let G be a finitely generated torsion-free group with a normal abelian subgroup A of finite index. If G/A is cyclic, then it was shown in Formanek (1970 [24]) that G has a finite subnormal series with infinite cyclic quotients and therefore Theorem 10.3 applies here. The next simplest case occurs with G/A a fours group. Here we have a rather interesting example, namely the group

$$G = \langle x, y \,|\, x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} \rangle.$$

This group is indeed torsion free, $A = \langle x^2, y^2, (xy)^2 \rangle$, and G cannot be left-ordered since if $x > 1$ and $y^\varepsilon > 1$ for some $\varepsilon = \pm 1$ then $1 = (xy^\varepsilon)^2(y^\varepsilon x)^2 > 1$, a contradiction. It is surprising how difficult it is to handle this group. If char $K \neq 2$, then $K[G]$ has a ring of quotients which is the tensor product of two quaternion algebras but conditions for this to be a division ring are just too unpleasant to be useful. It turns out, however, that G is supersolvable and such groups can now be effectively studied.

A ring R is said to be an n-fir if all n-generator right ideals of R are free of unique rank. For our purposes we need the following.

Lemma 10.6. *The ring R is a 1-fir if and only if R has no proper zero divisors.*

To see this, suppose first that R is a 1-fir and $a \in R$, $a \neq 0$. Then $aR \simeq R/I$ as R-modules where I is the right annihilator of a and, since aR is free, we have $R \simeq I + aR$. Furthermore I is then a homomorphic image of R so I is a 1-generator ideal of R and hence is also free. We therefore have R written as a sum of free R-modules and by the uniqueness of rank we conclude that $I = 0$ and that R has no zero divisors. Conversely, if R has no proper divisors of zero then certainly for each $a \in R$, $a \neq 0$, the right ideal aR is a free R-module of rank 1. Furthermore, if $aR = \sum a_iR$ is another representation of this ideal as a free R-module with

$a_i \neq 0$ then $a_i = ab_i$, $a = \sum_j a_j c_j$ for suitable $b_i, c_i \in R$ so $a_i = \sum_j a_j c_j b_i$. Thus for $i \neq j$ we have $a_j c_j b_i = 0$. Since R has no zero divisors, it follows easily that only one summand occurs and the uniqueness is proved. We now quote

THEOREM 10.7 (Cohn 1968 [13]). *Let R_1 and R_2 be two n-firs containing a common division ring D. Then the free product of R_1 and R_2 amalgamating D is also an n-fir.*

As an application of the above to group rings we have

THEOREM 10.8 (Lewin 1972 [57]). *Let $G = A *_N B$ be the free product of groups A and B amalgamating the common normal subgroup N. If $K[A]$ and $K[B]$ have no proper zero divisors and if $K[N]$ satisfies the Ore condition, then $K[G]$ has no proper divisors of zero.*

Normally, free products of groups are quite complicated in nature but the dihedral group $Z_2 * Z_2$ is an exception. Now it is quite easy to see that any infinite supersolvable group G has a normal subgroup N with $G/N \simeq Z$ or $Z_2 * Z_2$. In the latter case if A and B are the subgroups of G containing N which correspond to the two Z_2 factors then $G = A *_N B$ and the above applies. With this observation, we obtain the following result by induction on the torsion-free rank of G.

THEOREM 10.9 (Formanek 1973 [29]). *Let G be a torsion-free, supersolvable group. Then $K[G]$ has no proper divisors of zero.*

We remark that A. E. Zalesskii (1975 [123] and R. T. Bumby have independently discovered elementary proofs of Theorem 10.8 for the special case of $G/N = Z_2 * Z_2$. Thus Theorem 10.9 now has a fairly simple proof independent of n-firs. It would of course be nice to extend this theorem to polycyclic groups. However the only general result here is Theorem 2.9.

Finally there is the question of embedding $K[G]$ in a division ring and again Theorem 10.7 plays a role. A ring R is said to be a semifir if it is an n-fir for all n, and R is a fir if all right ideals of R are free of unique rank. Then another deep result of Cohn (1972 [14]) states that semifirs can always be embedded in division rings. Hence, for example, if G is a nonabelian free group then $K[G]$ can be embedded in a division ring but of course not as an order in it. This was proved originally by Malcev (1948 [112]) and B. H. Neumann (1949 [113]). Additional results of interest on this problem can be found in (Hughes 1970 [43], 1972 [44]). We remark that G. M. Bergman [6] has generalized Theorem 10.7 to firs and hence we conclude that the group ring of a free group is always a fir.

11. Dimension subgroups

Perhaps the oldest of the group ring problems is the one on dimension subgroups. If I is an ideal of $K[G]$ we set $G(I) = \{x \in G \mid x - 1 \in I\}$. We then have

LEMMA 11.1. *$G(I)$ is a normal subgroup of G. Furthermore, if I_1 and I_2 are ideals of $K[G]$ with $I_1 I_2 = I_2 I_1$ then $(G(I_1), G(I_2)) \subseteq G(I_1 I_2)$.*

Most of the above is obvious and only the last part requires proof. Here f $x \in G(I_1)$, $y \in G(I_2)$ then

$$x^{-1}y^{-1}xy - 1 = x^{-1}y^{-1}((x-1)(y-1) - (y-1)(x-1)) \in I_1 I_2$$

so the result follows.

The dimension subgroups $D^n(K[G])$ are defined by $D^n(K[G]) = G(\omega(K[G])^n)$. As it turns out, these normal subgroups depend on the characteristic of K but not on the field itself. Let $\gamma^n(G)$ denote the nth term of the lower central series for G so that $\gamma^1(G) = G$ and $\gamma^n(G) = (\gamma^{n-1}(G), G)$. Then $G/\gamma^n(G)$ is nilpotent and it is trivial to see that the set of torsion elements in such groups form characteristic subgroups. Thus $\delta_0^n(G) = \sqrt{(\gamma^n(G))}$ is a characteristic subgroup of G. On the other hand, if $p > 0$ is a prime we define the groups $\delta_p^n(G)$ inductively by $\delta_p^1(G) = G$ and $\delta_p^n(G)$ is generated by $(\delta_p^{n-1}(G), G)$ and all pth powers of elements in $\delta_p^{(n/p)}(G)$, where (n/p) is the smallest integer $\geqq n/p$.

LEMMA 11.2. *If* char $K = 0$ *then $D^n(K[G]) \supseteq \delta_0^n(G)$ and if* char $K = p > 0$ *then $D^n(K[G]) \supseteq \delta_p^n(G)$.*

To see this let us first assume that char $K = 0$. Then since $D^1(K[G]) = G$, Lemma 11.1 and induction yield easily $D^n(K[G]) \supseteq \gamma^n(G)$. Now if $x^t \in \gamma^n(G) \subseteq \gamma^{n-1}(G)$ then by induction $x - 1 \in \omega(K[G])^{n-1}$ so $(x-1)^2 \in \omega(K[G])^n$. Thus from

$$x^t - 1 = \sum_1^t \binom{t}{i} (x-1)^i$$

we conclude that $t(x-1) \in \omega(K[G])^n$ and hence since char $K = 0$, $x \in D^n(K[G])$. Now let char $K = p > 0$. Then again Lemma 11.1 and induction yield $D^n(K[G]) \supseteq (\delta_p^{n-1}(G), G)$. Furthermore, if $x \in \delta_p^{(n/p)}(G)$ then $x - 1 \in \omega(K[G])^{(n/p)}$ so

$$x^p - 1 = (x-1)^p \in \omega(K[G])^{(n/p)p} \subseteq \omega(K[G])^n$$

and the result follows.

Now the inclusions in the above lemma are in fact equalities. To prove this we need the following two crucial reductions due to P. Hall (1957 [35]) in characteristic 0 and G. Higman in characteristic p (see for example (Passi 1968 [114]) and (Bovdi 1969 [10]). First we may assume that G is finitely generated since if $x \in D^n(K[G])$ then there exists a finitely generated subgroup H of G with $x \in D^n(K[H])$. Moreover if the latter implies that $x \in \delta_a^n(H)$ for $a = 0$ or p then we can conclude that $x \in \delta_a^n(G)$ since $\delta_a^n(H) \subseteq \delta_a^n(G)$. Secondly, we may assume that $\delta_a^n(G) = \langle 1 \rangle$ since if $\bar{G} = G/\delta_a^n(G)$ then $\delta_a^n(\bar{G}) = \langle 1 \rangle$ and by Lemma 11.2, $D^n(K[\bar{G}]) = D^n(K[G])/\delta_a^n(G)$. By combining these two observations we may therefore assume that G is a finitely generated, torsion-free, nilpotent group if char $K = 0$ and that G is a finite p-group if char $K = p$. At this point the hard work begins and it is very technical and computational. The main results are the following.

THEOREM 11.3 (Jennings 1955 [49]). *If* char $K = 0$ *then* $D^n(K[G]) = \delta_0^n(G)$.

THEOREM 11.4 (Jennings 1941 [48]). *If* char $K = p > 0$ *then* $D^n(K[G]) = \delta_p^n(G)$.

A second proof of the latter can be found in (Lazard 1953 [56]). Furthermore Lazard offers an alternate group theoretic characterization of $\delta_p^n(G)$ which is more readily generalized to handle coefficient rings of prime power characteristic. A third and novel proof can be found in (Quillen 1968 [121]). Finally a Lie analog of the above exists and can be found in (Passi-Sehgal [115]).

The problem that remains is to find the dimension subgroups of the integral group ring. It was thought for a while that here $D^n(Z[G]) = \gamma^n(G)$, but a recent counterexample of Rips (1972 [84]) has negated this. Nevertheless a number of positive results do exist as can be seen from the discussion in (Gruenberg 1970 [110]).

Finally, there is a related question which concerns the nature of the intersection $\bigcap_{n=1}\omega(Z[G])^1$ and of certain transfinite analogs. Of course one can also consider $K[G]$ here but it makes more sense to study $Z[G]$ because of its relationship to nilpotence properties of wreath products. Some interesting results along this line can be found in (Baumslag 1959 [4]), (Hartley 1970 [38]) and (Gruenberg-Roseblade 1972 [33]).

12. Group isomorphisms

There is, finally, one rather natural question to consider, namely to what extent does $K[G]$ determine G. This turns out to be very difficult to answer and progress has so far been limited to groups which are either finite or abelian. Suppose first that G is finite. If $K = C$ is the field of complex numbers then $C[G] \simeq \sum C_{n_i}$ is a direct sum of full matrix rings over C and hence $C[G]$ only determines the set of degrees n_i. Thus, for example, if G and H are abelian of the same order then $C[G] \simeq C[H]$. It therefore makes more sense to consider the prime fields Q and $GF(p)$ and in fact we have

THEOREM 12.1 (Passman 1965 [65]). *Let G and H be finite groups and suppose that $Q[G] \simeq Q[H]$. Then for all fields K whose characteristic is prime to $|G| = |H|$ we have $K[G] \simeq K[H]$.*

Thus, for example, if G and H are the two nonabelian p-groups of order p^3 with $p > 2$ then $K[G] \simeq K[H]$ for all fields K of characteristic unequal to p. On the other hand, $GF(p)[G] \not\simeq GF(p)[H]$ here. It is therefore apparent that the fields of interest for this problem are those $GF(p)$ with $p \mid |G|$. Indeed, if G is a p-group then it is quite possible that $GF(p)[G] \simeq GF(p)[H]$ always implies $G \simeq H$. Some evidence along this line can be found in (Passman 1965 [66]) where techniques of Jennings (1941 [48]) were used to handle p-groups of order $\leqq p^4$.

Probably the most exciting result here is

THEOREM 12.2 (Dade 1971 [17]). *There exist two nonisomorphic finite groups G_1 and G_2 such that $K[G_1]$ and $K[G_2]$ are isomorphic for all fields K.*

To construct these groups we choose primes p and q with $q \equiv 1 \bmod p^2$, so that q is, of course odd. If Q_1 and Q_2 are the two nonabelian groups of order q^3 then since $p \mid q-1$ each group admits an appropriate automorphism of order p, acting nontrivially on its center. Thus if $\langle x_1 \rangle$ is cyclic of order p^2 and $\langle x_2 \rangle$ is cyclic of order p, then these groups act naturally on Q_i with x_1^p acting trivially. We then set

$$G_1 = Q_1\langle x_1 \rangle \times Q_2\langle x_2 \rangle$$

and

$$G_2 = Q_2\langle x_1 \rangle \times Q_1\langle x_2 \rangle.$$

To see that these groups are not isomorphic we consider H_i, the set of qth powers of elements of G_i'. Then H_i is a subgroup of order q here and $C_{G_1}(H_1)$ and $C_{G_2}(H_2)$ have nonisomorphic Sylow p-subgroups.

Now, if char $K \neq q$ then it is really not surprising that $K[G_1] \simeq K[G_2]$. Indeed, by a close look at the structure of the group rings, it is shown that

$$K[Q_1\langle x_i \rangle] \simeq K[Q_2\langle x_i \rangle]$$

for $i = 1, 2$ which clearly yields the required isomorphism. On the other hand, if char $K = q$ then the result is surprising and all the more surprising is how easily it is proved. Let char $K = q$ so that $K \supseteq GF(q)$. Since $p \mid q-1$, K contains a nonidentity pth root of unity ε and we can define

$$e_k = \frac{1}{p} \sum_{j=0}^{p-1} (\varepsilon^k x_1^p)^j \in K[G_i]$$

for $i = 1, 2$ and $k = 0, 1, \cdots, p-1$. These are, in fact, orthogonal central idempotents, since $\langle x_1^p \rangle$ is central in G_i, and hence we have the ring direct sums

$$K[G_i] = \sum_{j=0}^{p-1} e_k K[G_i].$$

Now each of these summands is a twisted group ring of $G_i/\langle x_1^p \rangle$ with $e_0 K[G_i] \simeq K[G_i/\langle x_1^p \rangle]$ and normally there is no relationship among these. However, here the subgroup $\langle x_1^p \rangle$ also exists as part of the abelian quotient

$$G_i/(Q_1 \times Q_2) \simeq \langle x_1 \rangle \times \langle x_2 \rangle$$

and more can be deduced. Indeed, if we use for the first time the full assumption $p^2 \mid q-1$, then K contains a p^2 root of unity δ with $\delta^p = \varepsilon$ and we can define linear characters $\lambda_k \colon G_i \to K$ satisfying $\lambda_k(x_1) = \delta^k$ so that $\lambda_k(x_1^p) = \varepsilon^k$. These characters then give rise, as in Section 3, to automorphisms $\tilde{\lambda}_k$ of $K[G_i]$. Since clearly $\tilde{\lambda}_k(e_0) = e_k$ it follows that for each i all the summands $e_k K[G_i]$ are isomorphic. Thus finally $e_k K[G_1] \simeq e_k K[G_2]$ since $G_1/\langle x_1^p \rangle \simeq G_2/\langle x_1^p \rangle$ and the theorem is proved.

There are a large number of results on finite abelian groups and we mention only two of these.

THEOREM 12.3 (Perlis-Walker 1950 [81]). *If G is a finite abelian group then $Q[G] \simeq Q[H]$ implies $G \simeq H$.*

THEOREM 12.4 (Deskins 1956 [18]). *If G is a finite abelian p-group and* char $K = p$ *then $K[G] \simeq K[H]$ implies $G \simeq H$.*

For finite groups, the problem of real interest is whether the integral group ring $Z[G]$ determines G. Here the best results can be found in Whitcomb's thesis (1967 [95]) which offers an affirmative answer in the case of metabelian groups (see also Jackson 1969 [111])). In particular, this shows that for Dade's groups, $Z[G_1] \not\simeq Z[G_2]$. The earlier examples of Dade (1964 [16]) are also of interest.

Suppose now that G is infinite. Then there is one obvious situation in which $K[G]$ determines G, namely when G is a tup-group or when perhaps for some other reason we know that all units in $K[G]$ are trivial. In this case $U = U(K[G])$, the group of units in $K[G]$, is just $U = \{kg \mid k \in K, k \neq 0, g \in G\}$ and G is of course a certain subgroup which we wish to identify. Let $\lambda: K[G] \to K$ be any K-homomorphism, for example, $K[G] \to K[G/G] \simeq K$, and set $U_\lambda = \{u \in U \mid \lambda(u) = 1\}$. Then the map $G \to U_\lambda$ given by $g \to \lambda(g)^{-1}g$ yields an isomorphism of G with U_λ and hence we have easily

THEOREM 12.5. *Suppose $K[G]$ and $K[H]$ are K-isomorphic and that all units in $K[G]$ are trivial. Then $G \simeq H$.*

The most extensive study of this problem for countably infinite abelian groups is due to Berman. However, since many of his results are extremely technical in nature we will only offer rough statements here.

THEOREM 12.6. (Berman 1967 [7], [8]). *Let G and H be countable abelian p-groups and let* char $K = p$. *Then $K[G] \simeq K[H]$ if and only if $G \simeq H$.*

Suppose now that char $K \neq p$ and let ε_i denote a primitive p^ith root of unity over K. We say that K is of the first kind (with respect to p) if $K(\varepsilon_j) \neq K(\varepsilon_2)$ for some $j > 2$. Otherwise K is of the second kind.

THEOREM 12.7 (Berman 1967 [7], [8]). *Let p be a fixed prime and let K be a field of the first kind. Then all countably infinite abelian p-groups fall into one of nine types and if G and H are of different types then $K[G] \not\simeq K[H]$. Furthermore, if G and H are of the same type then the possible occurrences of $K[G] \simeq K[H]$ can be enumerated.*

THEOREM 12.8 (Berman 1967 [7], [8]). *Let p be a fixed prime and let K be a field of the second kind. If either $p > 2$ or $p = 2$ and $\varepsilon_2 \in K$ then $K[G] \simeq K[H]$ for all countably infinite abelian p-groups G and H.*

Furthermore the case of $p = 2$ and $\varepsilon_2 \notin K$ can also be completely described but it is more complicated. Finally we have

THEOREM 12.9 (Berman 1967 [7], [8]). *Let G and H be countably infinite periodic abelian groups and let K be an algebraically closed field of characteristic prime to the orders of the elements of G and H. Then $K[G] \simeq K[H]$.*

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UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN, U.S.A.

AND

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA, U.S.A.